Reconstructing bifurcation diagrams from noisy time series using nonlinear autoregressive models

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We introduce a formalism for the reconstruction of bifurcation diagrams from noisy time series. The method consists in finding a parametrized predictor function whose bifurcation structure is similar to that of the given system. The reconstruction algorithm is composed of two stages: *model selection* and *bifurcation parameter identification*. In the first stage, an appropriate model that best represents all the given time series is selected. A nonlinear autoregressive model with polynomial terms is employed in this study. The identification of the bifurcation parameters from among the many model parameters is done in the second stage. The algorithm works well even for a limited number of time series. [S1063-651X(99)12607-2]

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An analysis of experimentally measured time series has been used to gain insights into the underlying physical processes, to do prediction, as well as to determine invariants associated to the dynamics, such as Lyapunov exponents and correlation dimension, among others. For reviews, see [1,2].

When time series measured at different values of the system parameters are given, additional information about the system's behavior becomes available. This extra information can be exploited to reveal the different bifurcations the system undergoes as the parameters are changed, as well as to uncover behaviors of the system which may be present but not readily observed.

The goal of bifurcation diagram (BD) reconstruction is to address this problem by obtaining a BD qualitatively similar to that of the given system using time series measured at a finite number of parameter values. In this reconstruction problem, the equations governing the dynamics of the system are unknown. Instead, time series at different parameter values are used in the reconstruction. The values, or even the number of parameters, may be unknown. These assumptions make the BD reconstruction problem formidable. This is because available methods to analyze bifurcation structures often require *a priori* knowledge of the dynamical equations [3] which can prove difficult to construct even for simple systems.

But recently, the BD reconstruction problem has received considerable attention [4–7]. This was brought about by the development of new algorithms for estimating predictor functions at fixed parameter values and the increasing need to characterize the different behaviors of systems with unknown dynamics using observations.

In this Brief Report, we describe an algorithm for reconstructing BDs from time series even when these are corrupted by observation and dynamical noise. This algorithm is divided into two stages: *model selection* and *bifurcation parameter identification*. In the first stage, we select an appropriate model that best represents all the given time series. In the second stage, we identify which among the many model parameters correspond to the bifurcation parameters of the system. In the following, we detail consecutively the two stages of the BD reconstruction algorithm and describe their implementation through an example.

Model selection. At this stage, nonlinear autoregressive (NAR) models with polynomial terms are employed as predictor functions for each time series. This is motivated by the following: NAR models are particularly effective for modeling noisy time series; their dependence on the parameters, i.e., coefficients of the polynomial are linear, which makes the structure of the model simple; and most importantly, an efficient scheme to compute the model parameters exists. This scheme makes possible the construction of parsimonious models necessary in the BD reconstruction problem.

More precisely, we assume that K time series $S_i=\{y_0^i,y_1^i,\ldots,y_N^i\}$, $i=1,\ldots,K$ are measured at different parameter values of a given dynamical system. We are interested in finding predictor functions $g(X;\mathbf{a})$ of the form

$$\begin{aligned} y_n^{\text{pred}} &= g(Y_{n-1};\mathbf{a})+\epsilon_n \\ &= a_0+a_1y_{n-1}+\cdots+a_dy_{n-d}+a_{d+1}y_{n-1}^2 \\ &\quad +a_{d+2}y_{n-1}y_{n-2}+\cdots+a_My_{n-d}^k+\epsilon_n \qquad (1) \\ &= \sum_{m=0}^{M} a_m z_m(Y_{n-1})+\epsilon_n, \qquad (2) \end{aligned}$$

where $Y_{n-1}=(y_{n-1},\ldots,y_{n-d})$ represents a vector in the d-dimensional reconstructed state space, the functional basis $\{z_m(X)\}$ is composed of all the distinct combinations of the coordinates up to degree k, $\mathbf{a}=(a_0,\ldots,a_M)$ represents the parameter set, ϵ_n accounts for the random forcing of the system, and $M+1=(k+d)!/(d!k!)$ determines the number of coefficients to be computed.

In particular, we want to find $\mathbf{a}^1,\ldots,\mathbf{a}^K$ such that $g(X;\mathbf{a}^i)$ is a predictor function for the ith time series. Moreover, we want the predictors of all time series to have the same structure, i.e., the same terms to be present in all of them. To do this, Korenberg's algorithm [8] is used to get the optimal number of terms in Eq. (1) and to compute the values of their associated coefficients, that is, the set $\{\mathbf{a}^i\}$.

Bifurcation parameter identification. In general the model given by Eq. (1) has more parameters than the original sys-

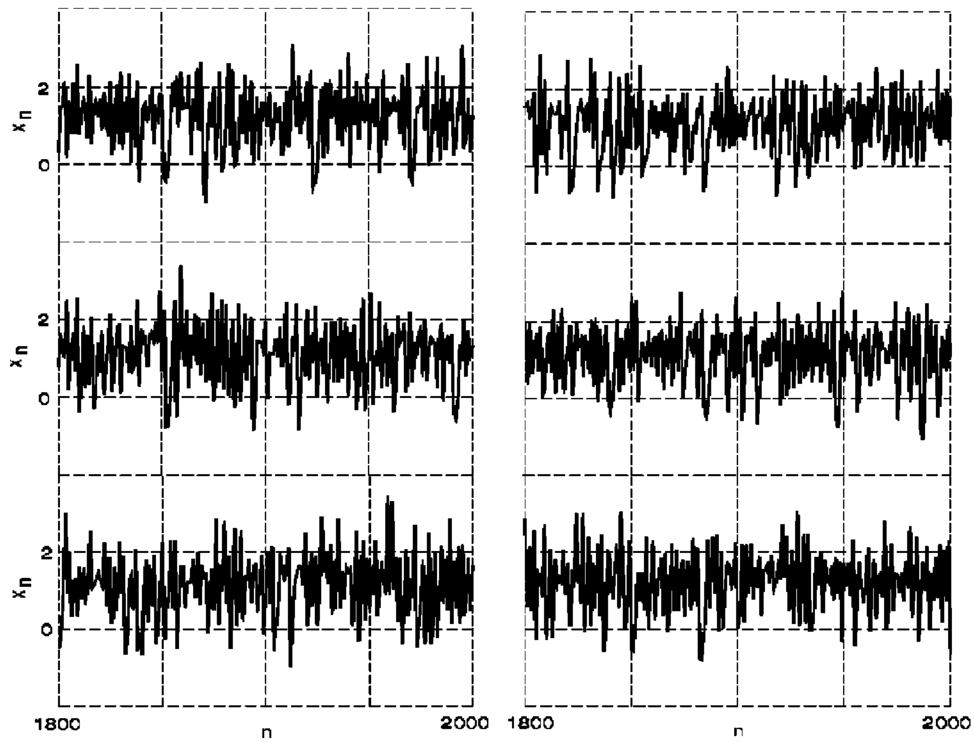

FIG. 1. Sample time series from the sine map [11]. Abscissae and ordinates are dimensionless.

tem. To determine the bifurcation parameters, the computed coefficients of each predictor function are collected to form the set of parameter vectors $\{\mathbf{a}^i=(a_0^i,\ldots,a_M^i)\}$, $i=1,\ldots,K$. Each $\mathbf{a}^i$ represents a point in the parameter space of the model. We call the region where these points are located the ''projection region.'' In other words, the projection region is the region in the parameter space of the model defined by the set of points $\{\mathbf{a}^i\}$ for all possible time series that can be generated by varying the bifurcation parameters of the system. For parameter values within this region, the dynamics of the model is therefore the same as that of the given system. Thus, one can take the BD of the model in this region as the reconstructed BD. The problem is to determine the projection region using the finite set of points $\{\mathbf{a}^i\}$ computed from the available time series.

This problem is related to the problem of finding lower-dimensional manifolds in a high-dimensional space which can be solved using several well-established approaches. Thus when the projection region is a nonlinear curve in the parameter space of the model, principal curves [9] can be employed to approximate this region. This approach is considered in [5] for one-dimensional BD reconstruction. For higher-dimensional cases, this approach is generalized as principal surfaces, nonlinear principal component analysis, and bottleneck neural networks (NNs), among others [9,10]. Alternatively, the method presented in [6] can also be used when the system is a map which is linear in parameter (LIP).

In many situations, the projection region is well approximated by a linear subspace of the parameter space of the model. This is the case when dealing with a small parameter region, reconstructing specific bifurcations, or the given system is a LIP map. Under these conditions, principal component analysis (PCA) provides a computationally efficient method to determine a satisfactory approximate of the projection region.

In the PCA-based approach, the covariance matrix of $\{\mathbf{a}^i\}$ is computed using

$$C=\frac{1}{K}\sum_{i=1}^{K}\delta\mathbf{a}_i\delta\mathbf{a}_i^T, \tag{3}$$

where $\delta\mathbf{a}_i=\mathbf{a}^i-\mathbf{m}$ and $\mathbf{m}$ represents the mean. The number of significant eigenvalues of C gives the dimension of the projection region and hence corresponds to the number of bifurcation parameters of the system. Moreover, the eigenvectors associated to the significant eigenvalues span the required projection region. Thus any point in this region can be expressed as

$$\mathbf{a}_{\mathrm{PR}}(\mu)=\mathbf{m}+\sum_{i=1}^{P}\mu_i\mathbf{e}_i, \tag{4}$$

where the $\mathbf{e}_i$'s are the eigenvectors associated with the P significant eigenvalues and $\mu=(\mu_1,\ldots,\mu_P)$ represent the expansion coefficients. The BD of the model $(X;\mu)\rightarrow g(X;\mathbf{a}_{\mathrm{PR}}(\mu))$ on the projection region can then be taken as the reconstructed BD with μ as the effective bifurcation parameter.

Implementation. We now illustrate the implementation of the algorithm through an example. Twenty time series, each of length $N=20\,000$, are given [11]. Some of these are shown in Fig. 1. A mere visual examination of the given time series does not reveal any significant difference in the behavior of the system. Our purpose is to distinguish the different behaviors by unraveling the putative bifurcations that separate them, as well as to uncover possible behaviors of the system not readily observed in the available time series. We recall that the system equations and the number of parameters (and therefore their values) are unknown.

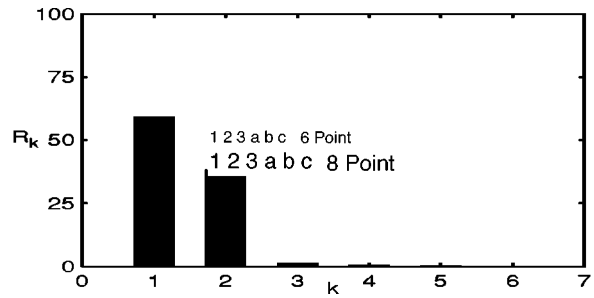

FIG. 2. Eigenvalues (in decreasing order) expressed in relative units $R_k = (\lambda_k / \Sigma_{i=1}^{7} \lambda_i) \times 100$ where λ_k represents an eigenvalue and k represents the order. Abscissas and ordinates are dimensionless.

At the first stage, we determine a model that accounts for the observed behaviors. To do this, we apply Korenberg's algorithm to obtain predictor functions for each time series. Comparison of the predictors shows that a NAR of the form

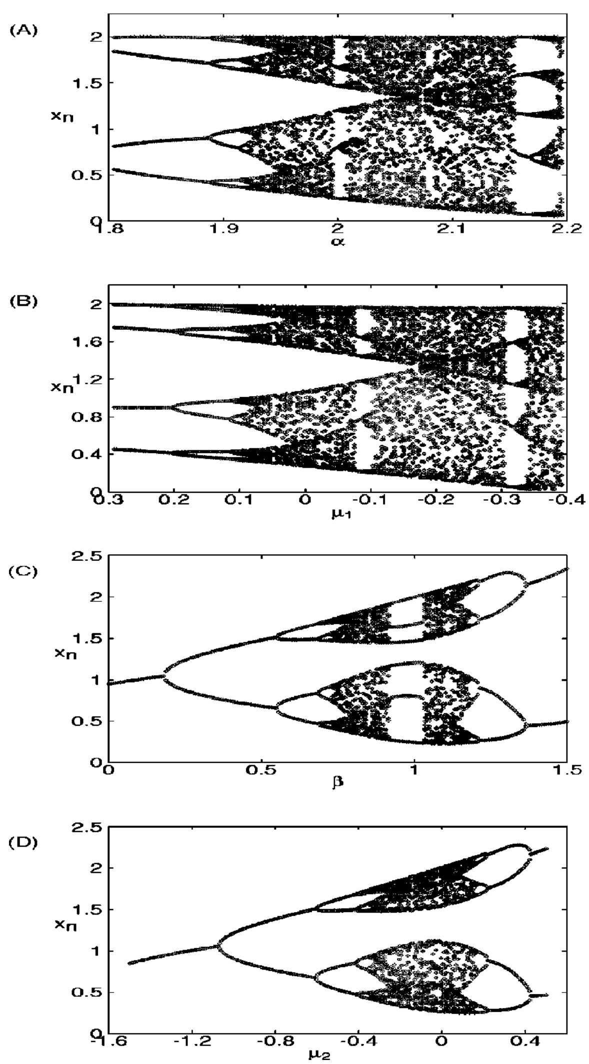

FIG. 3. (A) Original BD of the given system with α as the bifurcation parameter and (B) the reconstructed BD. (C) Original BD with β as the bifurcation parameter and (D) the reconstructed BD. Units: dimensionless.

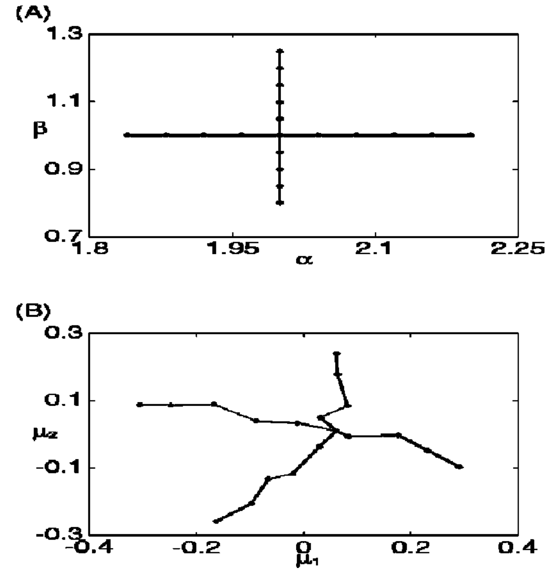

FIG. 4. (A) Parameter values used in generating the 20 time series. (B) Projections of the computed $\{\mathbf{a}^i\}$ onto the projection region spanned by the eigenvectors associated with the two significant eigenvalues. The x axis is spanned by the first eigenvector and the y axis is spanned by the second eigenvector. Units: dimensionless.

$$x_n = a_0 + a_1 x_{n-1} + a_2 x_{n-1}^2 + a_3 x_{n-1}^3 + a_4 x_{n-1}^4 + a_5 x_{n-1}^5 + a_6 x_{n-1}^6 + \epsilon_n \tag{5}$$

provides a suitable model for all the time series. This model has seven coefficients. Therefore, we dispose of 20 parameter vectors of order 7 from which to estimate the projection region. The PCA of this set of vectors yields two significant eigenvalues as shown in Fig. 2. This suggests that two system parameters were varied when the given time series were generated. This corresponds exactly to [11].

The projection region is now given by Eq. (4) with $P=2$, and $\mathbf{e}_1$ and $\mathbf{e}_2$ are the eigenvectors associated to the two significant eigenvalues. The effective bifurcation parameters are given by μ_1 and μ_2. One-dimensional BD reconstructions with respect to each parameter μ_1 and μ_2 are shown in Figs. 3(B) and 3(D), respectively. From the figure, it is evident that the BD of Eq. (5) is qualitatively the same as that of the original system shown in Figs. 3(A) and 3(C). The reconstructed two-dimensional BD (not shown) on the projection region also captures the different dynamics of the original system.

To show the correspondence between the two parameter sets, $\{\alpha,\beta\}$ and $\{\mu_1,\mu_2\}$, the projections of $\{\mathbf{a}^i\}$ onto the projection region are plotted in Fig. 4(B). These projections are computed by taking the inner product between $\{\mathbf{a}^i\}$ and the two eigenvectors $\mathbf{e}_1$ and $\mathbf{e}_2$. The figure shows that the distribution of these points follows that of the original parameters $\{\alpha,\beta\}$.

Figure 4 also illustrates the effects of noise in the reconstruction process. From a computational point of view, noise generally corrupts the estimation of the model parameters. This would affect the location of the projected points in the parameter space, which, in turn, would affect the determination of the projection region. The proposed algorithm is robust to such effects as illustrated in Fig. 4, which shows similar distribution of the two parameter sets in spite of the presence of strong dynamical noise.

The performance of the algorithm as described in the preceding paragraphs was also demonstrated in several dynamical systems. The algorithm worked equally well with the Hénon map, the cubic map, the logistic map, and the FitzHugh-Nagumo equations. For systems described by polynomial equations, the algorithm determined the correct terms in the polynomial. For the continuous system, the algorithm preserved the different bifurcations of the given system in the reconstructed BD. We plan to present the detailed reconstruction of the BDs of other examples in the future.

The proposed algorithm is also applicable using other models as predictor functions. The requirement is that the model should be a universal approximator to ensure the existence of projection regions for any dynamical system. Thus, the algorithm also works well with neural networks [4–6]. However, the NN-based approach has several shortcomings such as the difficulty in handling time series corrupted with dynamical noise, among others.

The use of NAR models in this reconstruction algorithm is more advantageous than the NN-based approach. The efficacy of the NAR model has been demonstrated in a variety of problems, particularly in the analysis of noisy time series [8,12,13]. The NAR model has been applied effectively in obtaining predictor functions for a number of systems (maps and flows), detecting nonlinearities in noisy time series (observation and dynamical noise), and estimation of dynamical invariants, among others. Furthermore, NAR models with appropriate number of terms can also capture bifurcation structures as shown in [13]. Aside from polynomials, NAR models can also have other basis functions.

The algorithm is also computationally efficient in a number of ways. In obtaining predictor functions using Korenberg's scheme, the problem of multiparameter optimization is eliminated by employing auxiliary polynomials which are orthogonal with respect to the natural invariant measure of the time series [1,14]. With this, the parameters are readily obtained from the time series. This scheme also leads to robust-to-noise estimation of the parameters since no distances in the reconstructed state space need to be computed. Moreover, the construction of parsimonious models becomes possible since the contribution of each orthogonal term in reducing the error function can be computed from the time series.

In summary, we present an algorithm in reconstructing BDs from noisy time series. The algorithm consists in finding a parametrized predictor function whose bifurcation structure is qualitatively similar to that of the given system. To account for the effects of noise, nonlinear autoregressive models are used as predictor functions. The use of Korenberg's algorithm makes possible the construction of parsimonious models, which is advantageous in the reconstruction problem. The algorithm is robust to noise, making it more suitable when dealing with noisy time series. Moreover, the algorithm also works well even for a limited number of time series (20 for the example considered).

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